Irreducible characteristic cycles for orbit closures of a symmetric subgroup

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Characteristic cycles of K-orbit closures

Preliminaries

- $G = GL(n)$ (all groups over $\mathbb{C}$)
- $B \supset T$ standard Borel subgroup and maximal torus
- $X = G/B$ the flag variety
- $K = GL(p) \times GL(q)$
- V = K-orbits on X (finite set)
- $v \in V \leftrightarrow X_v = \overline{K\dot{v}B}/B = G_v/B$
- Here $G_v = \overline{K\dot{v}B} \subset G$

Main result

Theorem (GJL)

Certain K-orbit closures admit

$$\mu : Z_v \to X_v,$$

which is a small resolution with smooth and strongly reduced fibers. As a consequence, the characteristic cycle $CC(IC_{X_v})$ is irreducible.

This theorem, combined with a result of Jones, implies:

Corollary

The Chern-Mather class of X_v is given by

$$c_M(X_v) = \mu^*(c(Z_v) \cap [Z_v]).$$

Because we have a torus action with finitely many fixed points, and Z_v is smooth, this means the CM class of X_v is effectively computable.

A Positivity Conjecture

Using the above results, we have obtained some limited computational evidence for the following conjecture.

Conjecture

When expressed in terms of the Schubert basis, the coefficients of $c_M(X_v)$ *are positive. Moreover, this extends to equivariant positivity of the equivariant Chern-Mather class* $c_M^T(X_v) = \mu_*(c^T(Z_v) \cap [Z])$.

Motivation (following Jones)

- Let M be smooth and stratified: $M = \cup S_i$.
- Let $X = \overline{S_{i_0}}$ be the closure of a stratum.
- $IC^{\cdot}_X$ = intersection cohomology sheaf (complex of sheaves of $\mathbb{C}$-vector spaces)
- This is a constructible sheaf on M.

Any complex $\mathcal{F}$ on M which is constructible with respect to this stratification gives

$$CC(\mathcal{F}) = \sum_i c_i(\mathcal{F})[\overline{T^*_{S_i}M}],$$

a Lagrangian cycle on T^*M.

- $c_i(\mathcal{F})$ is an integer called the "microlocal multiplicity".

Chern classes

If X is smooth, $c(TX) \cap [X] \in A_*X$ (the Chow ring). Generalizations to possibly singular X:

(1) Chern-Schwartz-MacPherson class $c_{SM}(X)$: There is a unique map

$$c_* : \{\text{constructible functions on } X\} \to A_*X$$

such that

1. $c_*(\mathbf{1}_X) = c(TX) \cap [X]$ if X is smooth.
2. c_* commutes with proper pushforward.

By definition,

$$c_{SM}(X) = c_*(\mathbf{1}_X).$$

Chern classes

(2) Chern-Mather class $c_M(X)$: Let $\nu : \widetilde{X} \to X$ be the Nash blowup. $\widetilde{X}$ need not be smooth, but it has a bundle $\underline{S}$ which roughly plays the role of the tangent bundle. By definition,

$$c_M(X) = \nu_*(c(\underline{S}) \cap [\widetilde{X}]).$$

These invariants are all related to each other. For example,

$$c_*(\mathbf{1}_{\overline{S_i}}) = \sum_{i,j} f_{ij} c_M(\overline{S_j}),$$

where $(f_{ij})^{-1}$ is the "local Euler obstruction matrix".

Resolutions and CM classes

Chern-Mather classes are usually harder to calculate than CSM classes because they do not have the functorial properties of CSM classes.

However, Jones proved:

Theorem (Jones)

Let $X \subset M$ be as above (so $X = \overline{S_{i_0}}$). Assume:

(A) X admits a small resolution $\mu : Z \to X$.

(B) The characteristic cycle of X is irreducible, i.e., $CC(IC^{\bullet}_X) = [\overline{T^*_{S_{i_0}} M}]$.

Then

$$c_M(X) = \mu_*(c(TZ) \cap [Z]).$$

Hypotheses (A) and (B)

Hypotheses (A) and (B) are satisfied for

- Schubert varieties in the Grassmannian.
- Some other Schubert varieties in classical types.

Jones did not know of any other examples.

Irreducible characteristic cycles

We prove the following general result, used in our result about K-orbit closures.

Theorem (GJL)

If $\mu : Z \to X$ is a small resolution with smooth and strongly reduced fibers, then (B) holds.

Strongly reduced: For $x \in X$, $F = \mu^{-1}(x)$, then for any $z \in F$,

$$\ker d\mu_z = T_z F.$$

The containment $\supseteq$ always holds.

K-orbits and closures

We need more notation for a precise statement of our main theorem.

In general, G reductive, $\theta : G \to G$ an involution, $G_0^\theta \subset K \subset G^\theta$. For us:

- $G = GL(n)$
- θ = conjugation by $\begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$
- $K = G^\theta = GL(p) \times GL(q)$

Studied by Richardson-Springer (following Vogan, Lusztig-Vogan,....)

- W = Weyl group $\supset S = \{s_i\}$ simple reflections
- $I \subset S$ gives W_I with long element w_I
- Parabolic $P_I \supset B$

Bruhat ordering and monoid action

The set
$$V = \{\text{set of } K\text{-orbits}\}$$
has a "Bruhat" (closure) ordering.

Monoid action: Given $v \in V$, $s \in S$, define
$$v * s = \text{open orbit in } K\dot{v}P_s/B.$$

- Either $v * s > v$ or $v * s = v$.
- The braid relations hold.
- $v * s * s = v * s$.

Clans

For $G = GL(n) \supset K = GL(p) \times GL(q)$, all this has a well-known combinatorial description in terms of a set $V_{p,q}$ of "clans".

A clan in $V_{p,q}$ is a sequence $(a_1, \ldots, a_n)$, where each a_i is $+$, $-$, or a natural number, subject to:

- Each natural number occurs twice.
- $|+\ \text{signs}| + |\{\text{pairs of numbers}\}| = p$
- $|-\ \text{signs}| + |\{\text{pairs of numbers}\}| = q$

Two clans are considered the same if the signs agree and matching numbers occur in the same positions.

- For example, $(1 + 212)$ and $(2 + 121)$ are considered the same clan.

Clans and properties of orbits

Closed orbits: Each a_i is a sign.

Open orbits: Write $v_{p,q}^{max}$ for the clan corresponding to an open orbit. Then

$$v_{p,q}^{max} = \begin{cases} (1,2,\ldots,q,+,\ldots,+,q,\ldots,2,1) \text{ if } q \le p \\ (1,2,\ldots,p,-,\ldots,-,p,\ldots,2,1) \text{ if } p \le q. \end{cases}$$

For example, $v_{3,2}^{max} = (1,2,+,2,1)$.

Smooth orbit closures

Concatenation: One can concatenate $v^1 \in V_{p_1,q_1}$ and $v^2 \in V_{p_2,q_2}$ to get $(v^1, v^2) = (v^1|v^2) \in V_{p_1+p_2,q_1+q_2}$.

Example: $v^1 = (1+1) \in V_{2,1}$ and $v_2 = (2,3,3,2) \in V_{2,2}$, and

$$v = (v^1|v^2) = (1,+,1,2,3,3,2) \in V_{4,3}.$$

Smooth orbit closures: The orbit closure X_v is smooth if and only if

$$v = (v^1|v^2|\cdots v^r),$$

with $v^i = v^{max}_{p_i,q_i}$.

- The clan from the preceding example corresponds to a smooth orbit closure.

The monoid action

Let

$$v = (a_1, \ldots, a_n) \in V_{p,q}$$

and let s_i denote the i-th transposition. Then $v * s_i > v$ if and only if:

Case (1): At least one of a_i, a_{i+1} is a number, and one of the following occurs:

- a_i and a_{i+1} are unequal numbers, and the mate of a_i occurs before the mate of a_{i+1}.
- One of a_i, a_{i+1} is a sign and the other is a number whose mate appears on the same side of the sign.

In this case, $v * s_i$ differs from v by swapping a_i, a_{i+1}.

Case (2): a_i and a_{i+1} are opposite signs. In this case, $v * s_i$ differs from v by replacing these entries with a pair of equal numbers.

Resolutions

- Recall: $G_{v_0} = \overline{K\dot{v}_0 B}$ and $X_{v_0} = G_{v_0}/B$.
- If $v_0 * s > v_0$ then
$$G_{v_0} \times^B P_s/B \longrightarrow X_{v_0 * s}.$$
- For $G = GL(n) \supset K = GL(p) \times GL(q)$, this is birational, so if X_{v_0} is smooth, this is a resolution of singularities.
- If $v_0 < v_0 * s_{i_1} < \cdots < v_0 * s_{i_1} \cdots * s_{i_k} = v$, then we obtain a resolution
$$\mu : G_{v_0} \times^B P_{s_{i_1}} \times^B \cdots \times^B P_{s_{i_k}}/B \longrightarrow X_v.$$

Resolutions and commuting reflections

For our resolutions we assume:

- $I = \{s_{i_1}, \ldots, s_{i_k}\}$ is a set of commuting reflections
- $v_0 = (v^1, v^2, \cdots, v^r)$ with $v^i = v^{max}_{p_i, q_i}$
- $\ell(v * w_I) = \ell(v_0) + |I|$

The last condition implies that the s_i in I each straddle two blocks.

Under these assumptions,

$$P_I \cong P_{i_1} \times^B \cdots \times^B P_{i_k}/B,$$

so the resolution can be written

$$\mu : Z_v = G_{v_0} \times^B P_I/B \longrightarrow X_v$$

where $v = v_0 * w_I$.

Small resolutions

A resolution $f : X \to Y$ is small if

$$\operatorname{codim}\{y \in Y \mid \dim f^{-1}(y) \geq i\} > 2i.$$

In other words, the locus in Y where the fibers are large is small.

Theorem (Larson)

Let $\mu : Z_v \to X_v$ be as on the preceding slide. Assume that for all $y < v_0$ with $\ell(y) = \ell(v_0) - 1$, we have

$$\ell(y * w_I) = \ell(y) + |I|.$$

Then the resolution μ is small.

Small resolutions

Combinatorial formulation

The condition in the above theorem can be expressed combinatorially.

- None of the following four patterns occurs in v_0 for $s_i \in I$, where the bar is drawn between the entries in positions $i, i+1$ of v_0, ϵ is a sign, and a is a number:

$$\epsilon|aa, \quad aa|\epsilon, \quad \epsilon|a\epsilon, \quad \epsilon a|\epsilon.$$

Main result

The precise statement of our main result then takes the form:

Theorem (GJL)

Suppose X_{v_0} is a smooth orbit closure, and I is a set of commuting reflections such that $\ell(y * w_I) = \ell(y) + |I|$ for all $y \leq v_0$ with $y = v_0$ or $\ell(y) = \ell(v_0) - 1$. Then the resolution

$$\mu : Z_v = G_{v_0} \times^B P_I/B \to X_v,$$

has smooth and strongly reduced fibers.

Computing $c_M^T(X_v)$

Basic idea:

- Use localization on Z_v to compute

$$\mu_*(c(TZ) \cap [Z]) = \sum_{y \in W} b_y [yB] \in A_*(X) \otimes \mathcal{Q}$$

- It is well-known how to express $[yB]$ in the Schubert basis:

$$[yB] = \sum m_{yw}[Y_w],$$

so

$$c_M^T(X_v) = \sum_{y,w \in W} b_y m_{yw}[Y_w].$$

Weights on tangent spaces

- To use localization, we need to know the T-weights on tangent spaces to $Z_v = G_{v_0} \times^B P_I/B$.
- For this, we need the weights of tangent spaces on $X_{v_0} = \overline{Kv_0B}/B$.
- Problem: How to deal with taking closure?

Solution: Use the τ-invariant.

- By definition, $\tau(v_0) = \{s \in S \mid v_0 * s = s\}$.
- Consider $\pi : \overline{Kv_0B}/B \to Kv_0P_{\tau(v_0)}/P_{\tau(v_0)}$.
- The image of π is a closed orbit, so no need to take closures.
- $\overline{Kv_0B}/B = Kv_0P_{\tau(v_0)} \times^{P_{\tau(v_0)}} P_{\tau(v_0)}/B$.
- From this we can compute the weights on tangent spaces.

Example

Let

- $G = GL(4) \supset K = GL(2) \times GL(2)$.
- $v_0 = (1122), I = \{s_2\}, v = v_0 * s_2 = (1212)$.

We used $\mu : Z_v \to X_v$ to compute $c_M^T(X_v) = \mu_*(c^T(Z_v) \cap [Z_v])$, and expanded the result in terms of the Schubert basis $[Y_w]$ of $A_*^T(X)$:

$$c_M^T(X_v) = \sum_{w \in W} c_v^w [Y_w].$$

Example

The most complicated coefficient turns out to be c_v^{4321}.

Expressed as a sum of monomials in the simple roots α_i, we have

$$\begin{aligned}
c_v^{4321} =& \alpha_1^3\alpha_2^2\alpha_3 + \alpha_1^3\alpha_2^2 + \alpha_1^3\alpha_2\alpha_3^2 + 3\alpha_1^3\alpha_2\alpha_3 + 2\alpha_1^3\alpha_2 + \alpha_1^3\alpha_3^2 + 2\alpha_1^3\alpha_3 + \alpha_1^3 + 2\alpha_1^2\alpha_2^3\alpha_3 \\
&+ 2\alpha_1^2\alpha_2^3 + 3\alpha_1^2\alpha_2^2\alpha_3^2 + 9\alpha_1^2\alpha_2^2\alpha_3 + 6\alpha_1^2\alpha_2^2 + \alpha_1^2\alpha_2\alpha_3^3 + 7\alpha_1^2\alpha_2\alpha_3^2 + 12\alpha_1^2\alpha_2\alpha_3 \\
&+ 6\alpha_1^2\alpha_2 + \alpha_1^2\alpha_3^3 + 4\alpha_1^2\alpha_3^2 + 5\alpha_1^2\alpha_3 + 2\alpha_1^2 + \alpha_1\alpha_2^4\alpha_3 + \alpha_1\alpha_2^4 + 2\alpha_1\alpha_2^3\alpha_3^2 \\
&+ 7\alpha_1\alpha_2^3\alpha_3 + 5\alpha_1\alpha_2^3 + \alpha_1\alpha_2^2\alpha_3^3 + 9\alpha_1\alpha_2^2\alpha_3^2 + 16\alpha_1\alpha_2^2\alpha_3 + 8\alpha_1\alpha_2^2 + 3\alpha_1\alpha_2\alpha_3^3 \\
&+ 12\alpha_1\alpha_2\alpha_3^2 + 14\alpha_1\alpha_2\alpha_3 + 5\alpha_1\alpha_2 + 2\alpha_1\alpha_3^3 + 5\alpha_1\alpha_3^2 + 4\alpha_1\alpha_3 + \alpha_1 + \alpha_2^4\alpha_3 \\
&+ \alpha_2^4 + 2\alpha_2^3\alpha_3^2 + 5\alpha_2^3\alpha_3 + 3\alpha_2^3 + \alpha_2^2\alpha_3^3 + 6\alpha_2^2\alpha_3^2 + 8\alpha_2^2\alpha_3 + 3\alpha_2^2 + 2\alpha_2\alpha_3^3 + 6\alpha_2\alpha_3^2 \\
&+ 5\alpha_2\alpha_3 + \alpha_2 + \alpha_3^3 + 2\alpha_3^2 + \alpha_3
\end{aligned}$$

The coefficient of each monomial is positive.

This holds for all c_v^w for this v, confirming our conjecture in this case.